Key focus

This objective develops critical understanding of the assumptions underlying theoretical probability. Students explore the three core assumptions: fair devices (unbiased, uniform construction), equally likely outcomes (not just counting outcomes, but ensuring each has the same chance), and random processes (uncontrolled, unpredictable selection). The learning progresses from identifying when assumptions hold, to explaining why they matter, to recognising when real-world devices violate them and how that changes the model's accuracy. A key shift is moving beyond "all outcomes" counting to understanding that only equally likely outcomes allow the fraction method. Students also learn to interpret short-run variation and to use sample size as a lens for evaluating evidence of bias, rather than concluding that any deviation from theory signals a broken device.

Suggested teaching sequence

1. Introduce fair devices: coins, dice, spinners, well-shuffled decks. Ask: what makes them fair? Establish that fairness means each outcome has equal chance, not just that outcomes are possible. Discuss bias: a coin could be weighted, a die could be shaved, a spinner could be unbalanced.

2. Introduce equally likely outcomes: distinguish between counting outcomes and their probabilities. Example: a spinner with one large and two small sectors has 3 sectors but not equally likely outcomes. The fraction method $\frac{\text{favourable}}{\text{total}}$ only works when outcomes are equally likely.

3. Introduce randomness: explain that random means unpredictable and uncontrolled. If a player chooses deliberately or information influences the choice, it's not random.

4. Model finding probability using the fair-device assumption: "For a fair die, each face has probability $\frac{1}{6}$." Then ask: "What if the die is biased?" Lead to: we need extra information (the actual probabilities or experimental data).

5. Discuss small-sample variation: a fair coin can give 7 heads in 10 flips. This is not proof of bias; it's expected random variation. Compare 7 in 10 (70%) with 700 in 1000 (70%): the larger sample makes the same percentage much more suspicious.

6. Introduce replacement and independence: when drawing from a bag without replacement, the probabilities shift after each draw. This violates the independence assumption in many formula-based calculations. Contrast with replacement (probabilities stay the same).

7. Model critiquing probability claims: given a claim like "The game is fair because there are 8 equally sized sectors," ask students to identify implicit assumptions. What if the spinner is off-balance? What if it's spun with varying force? What if the pointer is biased toward one side?

8. Discuss experimental vs. theoretical probability: for a fair device, theory predicts; for a biased device, experiment reveals. If you cannot assume fairness, use data from many trials to estimate the actual probability.

9. Practise designing simple experiments to test fairness: how many trials? What would convince you a device is biased? Introduce the idea that very large deviations (e.g., 95 heads in 100 flips) are more evidence than small deviations (51 heads in 100 flips).

10. Explore common mistakes: gambler's fallacy ("heads is due after many tails"), confusing possible with likely, and assuming all claims of fairness without checking the device or experimental conditions.

Core examples to model

- Fair device, equal outcomes: A fair six-sided die: $P(6) = \frac{1}{6}$ because the die is fair and each face is equally likely.
- Assumption fails: A spinner with sectors of sizes 50%, 30%, 20%: You cannot say $P(\text{largest sector}) = \frac{1}{3}$. Outcomes are not equally likely, so you need to use the actual sector sizes: $P = 0.5$.
- Small vs. large sample: 8 sixes in 20 rolls = 40% (not proof of bias; a fair die gives 16.7% on average, but variation happens). 800 sixes in 2000 rolls = 40% (much more suspicious; very large samples usually match theory).
- Replacement matters: Bag with 3 red and 2 blue counters. With replacement: $P(\text{red}) = \frac{3}{5}$ every draw. Without replacement: first draw $P(\text{red}) = \frac{3}{5}$, second draw $P(\text{red}) = \frac{2}{4}$ or $\frac{3}{4}$ depending on the first outcome.
- Identifying hidden assumptions: "A random number generator gives each number 1 to 10 with equal probability." Assumptions: the generator is unbiased, each number is equally likely to be selected, the process is truly random (not influenced by previous numbers).
- Evaluating fairness: "This coin landed heads 18 times in 20 flips. Is it biased?" Discussion: 18/20 = 90%, well away from 50%. But small sample; a fair coin can produce this. More convincing would be 180 heads in 200 flips (still 90%, but larger sample). Compare to 51 heads in 100 flips (51%, barely different from 50%); this is easily explained by chance.

Watch for

- Assuming "possible" means "equally likely": rolling a die with 6 faces is possible, but only if all faces are equally likely do we use $\frac{1}{6}$. A weighted die makes outcomes possible but not equally likely.
- Misinterpreting small-sample variation: a fair coin can produce unusual results in 10 flips. This does not make the coin unfair; random variation is normal and expected.
- Ignoring the role of fairness in the probability formula: students may apply $\frac{\text{favourable}}{\text{total}}$ to any scenario. Remind them: this only works when the device is fair and outcomes are equally likely.
- Confusing experimental and theoretical probability: if a spinner is biased, theory (assuming fairness) will be wrong; experiment (spinning many times) reveals the truth. Students need both concepts.
- Underestimating the importance of sample size: small samples naturally vary widely. Use real examples to show how larger samples average out random variation and detect real bias more

reliably.

- Overlooking practical sources of bias: a physical die might be imperfect; a spinner might have friction on one side; a deck might be marked; a chosen "random" selection might be subconscious bias. Real-world devices often violate fairness.
- Misapplying the gambler's fallacy: after many heads, students might think tails is "due." Clarify: each flip is independent; past results do not change future probabilities (unless the device's properties have changed).

Checks for understanding

- Can students identify the three core assumptions of theoretical probability?
- Can they explain why equally likely outcomes matter for using the fraction method?
- Can they distinguish between a fair device and an unfair one, and describe what makes a device unfair?
- Can they explain why a coin giving 7 heads in 10 flips does not automatically prove bias?
- Can they use sample size to judge whether evidence of bias is convincing?
- Can they identify a hidden or unstated assumption in a probability claim?
- Can they evaluate a real-world probability scenario and spot where assumptions might fail?
- Can they design a simple experiment to test whether a device is fair?
- Can they distinguish between "possible" and "likely" in a probability context?
- Can they explain when to use experimental probability instead of theoretical probability?